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SINGULARITIES IN THE DYNAMICS OF SYSTEMS WITH NON-IDEAL CONSTRAINTS†

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The fundamental problem of dynamics involving determining the generalised accelerations and reactions of constraints as a function of the applied forces is considered for mechanical systems with $k \geqslant 1$ non-ideal geometrical constraints. A relation is established between this problem and the analysis of the singularities of piecewise-smooth mappings of a space R^k into itself. For Coulomb-type friction, a criterion for there to be no paradoxes is obtained and it is shown that when $k = 1$ possible singularities are convolutions, while when $k = 2$ they are subdivided into a fold, a casp and a double fold. The well-known Painlevé–Klein example is considered in detail for cases of bilateral and unilateral contacts; a complete list of possible paradoxical situations is presented for the first time. © 2003 Elsevier Science Ltd. All rights reserved.

Paradoxical situations of the non-existence or non-uniqueness of the solution in systems with one pair of frictions were discovered for the first time by Painlevé in 1895 [1]. A number of sufficient conditions for there to be no paradoxes were obtained in [2–6] in the case when $k \geqslant 2$.

1. CONSTRUCTION OF THE DEFINING MAPPING

We will introduce generalized coordinates $q_1, q_2, \dots, q_n$ such that the first $k \geqslant n$ of these are equal to the distances between those bodies between which contact is possible. Hence, the equation $q_j = 0$ $(j = 1, \dots, k)$ indicates the presence of a contact in the jth pair. We will assume that all these equalities are satisfied at the instant of time considered.

Bilateral of unilateral constraints between the bodies are possible depending on the constructive features. In the first case, the corresponding coordinate q_j is identically equal to zero, while in the second case it can take positive values. At the stage when setting up the equations of motion, we will validate the principle of constraint elimination and assume that the system has n degrees of freedom, without limiting the possible values of the coordinates. We will write the equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{k} R_i^{(j)}, \quad i = 1, 2, \dots, n$$
$$q_1 = \dots = q_l \equiv 0, \quad q_{l+1} \geq 0, \dots, q_k \geq 0 \tag{1.1}$$

where T is the kinetic energy of the system, Q_i are generalized forces and $R_i^{(j)}$ are the components of the reaction of the jth constraint (unilateral or bilateral).

We will solve the fundamental problem of dynamics involving determining the generalized accelerations and the reactions of the constraints. If the jth constraint were ideal, only the component $N_j = R_i^{(j)}$ would be non-zero. The presence of non-zero values of $R_i^{(j)} (i \neq j)$ indicates that there is friction. The friction law is described as the dependence of the components $R_i^{(j)} (i \neq j)$ on N_j for given values of the generalized coordinates and velocities

$$R_i^{(j)} = F_{ij}(\mathbf{q}, \dot{\mathbf{q}}, N_j), \quad j = 1, 2, \dots, k \tag{1.2}$$

In particular, for viscous friction the functions F_{ij} are independent of N_j, while for Coulomb sliding friction (we will assume the slipping velocity to be non-zero) they are linear with the respect to the modulus of the normal reaction

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$$F_{ij} = \alpha_{ij}(\mathbf{q}, \dot{\mathbf{q}})|N_j| \tag{1.3}$$

Substituting expressions (1.2) into Eqs (1.1), we obtain

$$\begin{gathered} \mathbf{C}\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{N} + \mathbf{F}(\mathbf{N}) + \boldsymbol{\Theta} \\ \mathbf{q} = (q_1, \ldots, q_n)^T, \ \mathbf{N} = (N_1, \ldots, N_k, 0, \ldots, 0)^T \end{gathered} \tag{1.4}$$

where $\mathbf{C} = \mathbf{C}(t, \mathbf{q})$ is a matrix, quadratic in terms of the velocities, of part of the kinetic energy of the system, $\mathbf{F}(\mathbf{N}) = \|F_{ij}\|$ is an $n \times n$ matrix, describing the friction, while the column vector $\boldsymbol{\Theta} = \boldsymbol{\Theta}(t, \mathbf{q}, \dot{\mathbf{q}})$ is made up of terms obtained on expanding the derivatives in Eqs (1.1) and not containing generalized accelerations.

In system (1.4) $\ddot{\mathbf{q}}$ and $\mathbf{N}$ are unknown. Note that the variables $\ddot{q}_j$ and N_j are not independent: for bilateral constraints $\ddot{q}_j \equiv 0$ ($j = 1, \ldots, l$) while for unilateral constraints the following conditions are satisfied [7, 8]

$$\ddot{q}_i \geq 0, \ N_i \geq 0, \ \ddot{q}_i N_i = 0; \ i = l+1, \ldots, k \tag{1.5}$$

Relations (1.5) enable us (after changing to dimensionless form in (1.4)) to associate a unique variable with the pair $\ddot{q}_j$, N_j by the formula

$$z_i = x_i - y_i, \ x_i = \ddot{q}_i, \ y_i = N_i; \ i = l+1, \ldots, k \tag{1.6}$$

where the unique pair $\ddot{q}_j$, N_i, satisfying conditions (1.5), corresponds to each value of z_i.

These considerations enable the number of unknowns in system (1.4) to be reduced to its order n. Further simplification of the system can be achieved by eliminating the quantities $\ddot{q}_s$ ($s = k+1, \ldots, n$) from it using the last $n - k$ equations. The equations finally obtained have a form similar to (1.4), but with changed expressions for **C**, **Q**, **F** and **Θ**. For clarity, we will keep the previous notation, assuming $n = k$ in (1.4).

Equations (1.4) give the mapping

$$\mathbf{Q} = \mathbf{S}(\mathbf{z}), \ \mathbf{S}(\mathbf{z}) = \mathbf{C}\ddot{\mathbf{q}} - \mathbf{N} - \mathbf{F}(\mathbf{N}) - \boldsymbol{\Theta}, \ z_j = N_j, \ \ddot{q}_j = 0; \ j = 1, \ldots, l \tag{1.7}$$

where the variables $z_{l+1}, \ldots, z_k$ are defined by expressions (1.6). The fundamental problem of dynamics being discussed reduces to transforming the mapping (1.7). The latter will be continuous if all the functions F_{ij} in formulae (1.2) are continuous with respect to N_j. However it does not follow from the differentiability of these functions when $l < k$ that **S** is differentiable, since the presence of unilateral constraints leads, in view of definitions (1.6), to a discontinuity when $z_i = 0$. Moreover, discontinuities on the surfaces $N_j = 0$ also correspond to bilateral constraints with dry friction of the form (1.3).

Hence, the defining mapping (1.7) in general is piecewise-smooth, and in the most important special case of dry friction it is piecewise-linear, with discontinuities in the coordinate planes of the space R^k. If each of the 2^k matrices of this mapping are non-degenerate, the number of solutions of the fundamental problem of dynamics lies in the range from 0 to 2^k, while in the case of degeneracy this number may be infinite. One can use the results in [9, 10] to check the non-degeneracy directly in system (1.4).

We will use a similar approach also in more complex cases when the friction is described by a law different from (1.2). In this case it may be necessary to consider the mapping (1.7) in a space of higher dimensionality than k. For example, Amonton's law describes dry friction between bodies at the instant when the velocity of relative slipping is equal to zero. In this case the friction forces depend not only on the normal reactions but also on the tangential accelerations. Hence, for a unique determination of **Q** from system (1.4) it is necessary to specify these accelerations or (when they are equal to zero) the friction forces. We will not discuss these situations in this paper.

2. THE CRITERION FOR THE EXISTENCE AND UNIQUENESS OF SOLUTIONS

In the case of dry friction (1.3) the mapping (1.7) takes the form

$$\mathbf{S}(\mathbf{z}) = \mathbf{C}\ddot{\mathbf{q}} - \mathbf{N} - \boldsymbol{\Phi}|\mathbf{N}| - \boldsymbol{\Theta}, \ |\mathbf{N}| = (|N_1|, |N_2|, \ldots, |N_k|)^T \tag{2.1}$$

where $\mathbf{\Phi} = ||\beta_{ij}(\mathbf{q}, \dot{\mathbf{q}})||$ is a square matrix of order n, the elements of which are obtained from (1.3) when reducing the order of system (1.4) from n to k.

We will represent the variables z_j ($j = 1, \ldots, l$) in the form

$$z_j = x_j - y_i,\ x_j = (|N_j| + N_j)/2,\ y_j = (|N_j| - N_j)/2;\ j = 1, \ldots, l \tag{2.2}$$

The variables x_j and y_j are obviously complementary, i.e. relations of the type (1.5) are satisfied for these. For values of the subscript $j = l + 1, \ldots, k$, corresponding to unilateral constraints, these variables are given by formulae (1.6).

We express the unknowns z_j in (2.1) in terms of x_j and y_j using (2.2). We finally obtain a linear algebraic system of k equations with k pairs of complementary variables

$$\begin{gathered}\mathbf{A}_1\mathbf{x} - \mathbf{A}_1\mathbf{y} = \mathbf{B},\ \mathbf{B} = \mathbf{Q} + \mathbf{\Theta},\ \mathbf{x}, \mathbf{y}, \mathbf{B} \in R^k,\ \mathbf{A}_{1,2} \in R^{k\times k} \\ \mathbf{x} \geq 0,\ \mathbf{y} \geq 0,\ \mathbf{xy} = \mathbf{0}\end{gathered} \tag{2.3}$$

Proposition 1. The fundamental problem of dynamics of finding the generalized accelerations and the reactions of the constraints in system (1.1) with dry friction (1.3) for specified values of the coordinates and velocities (the slipping velocities in contacts with friction are non-zero) for any applied forces has a unique solution if and only if all the principal minors of the matrix $\mathbf{A}^* = \mathbf{A}_1^{-1}\mathbf{A}_2$ in system (2.3) are positive.

The proof of this criterion for Eq. (2.3) was given previously in [11], and the relation between this equation and the main problem of dynamics was established in the previous section.

In practice, the evaluation of the matrix A^* can be reduced to expressing the variables $\mathbf{x}$ in terms of $\mathbf{y}$ from system (2.3). We will note some special cases of the solution of this problem.

1. If $l = 0$, i.e. all the constraints with friction have a unilateral form, then

$$\mathbf{A}_1 = \mathbf{C},\ \mathbf{A}_2 = \mathbf{E}_2 + \mathbf{\Phi}$$

Since $\mathbf{C}$ is symmetrical and positive-definite, to check the conditions of Proposition 1 we must convince ourselves that all the corner minors of the matrix $\mathbf{A}_2$ are positive.

2. If $l = k = 1$, i.e. there is a unique, albeit bilateral, constraint with friction, the criterion takes the form $A_1A_2 > 0$. Assume $\ddot{q} = 0$ in (2.1), we have $A_{1,2} = -1 \mp \Phi$, and hence this inequality implies that $|\Phi| < 1$.

Example. Consider a rod of length $2l$, the ends of which slide along two parallel straight lines (Fig. 1a). This system was first analysed by Painlevé [1] on the assumption that both constraints are bilateral and non-ideal. This example later became a popular model for demonstrating different ideas for overcoming the paradoxes in systems with friction (see [12–17]).

Here we will also consider a problem which allows of different forms of contact, i.e. by replacing one or both constraints by a unilateral constraint (in this case the rod can be situated in the space between the straight lines).

We will assume that the rod slides to the right at a given instant of time. We will put the mass of the rod equal to unity and write for it the theorem on the motion of the centre of mass and the theorem of moments

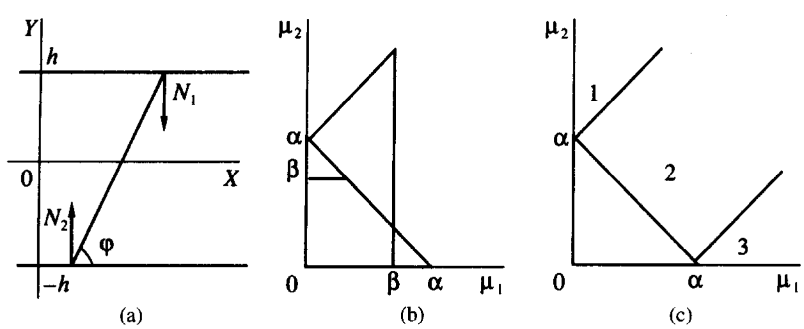

Fig. 1

$$\ddot{x} = -\mu_1|N_1| - \mu_2|N_2| + X, \quad \ddot{y} = -N_1 + N_2 + Y$$
$$k^2\ddot{\varphi} = h(\mu_1|N_1| - \mu_2|N_2|) - b(N_1 + N_2) + M, \quad b = l\cos\varphi \neq 0 \tag{2.4}$$

where $2l$ is the length of the rod, k is its radius of inertia, $2h$ is the distance between the directrices, x, y and φ are the coordinates of the centre of mass and the angle between the rod and the directrices, and X, Y and M are the applied forces and moment.

We will change from y, φ to the coordinates

$$q_1 = h - y - l\sin\varphi, \quad q_2 = h + y - l\sin\varphi$$

representing the distances from the ends of the rod to the directrices. At the instant of time considered $q_1 = q_2 = 0$, $\dot{q}_1 = \dot{q}_2 = 0$, whence it follows that $\dot{y} = 0$, $\dot{\varphi} = 0$. Taking Eqs (2.4) into account we obtain

$$\ddot{q}_i = (-1)^{i+1}(N_1 - N_2 - Y) + \frac{b^2}{k^2}(N_1 + N_2) - \frac{bh}{k^2}(\mu_1|N_1| - \mu_2|N_2|) - \frac{b}{k^2}M, \quad i = 1, 2 \tag{2.5}$$

We will consider different cases of the contacts.

1. Both constraints are bilateral, i.e. $q_1 \equiv 0$, $q_2 \equiv 0$. We determine the variables x_j, y_j ($j = 1, 2$) from formulae (2.2), and then solve system (2.5) for x_j, where the left-hand sides are replaced by zeros. We obtain

$$x_i = y_i + \kappa(\mu_1 y_1 - \mu_2 y_2) + \frac{\kappa}{2h}(M + (\mu_2 h + b)Y)$$
$$x_2 = y_2 + \kappa(\mu_1 y_1 - \mu_2 y_2) + \frac{\kappa}{2h}(M + (\mu_1 h + b)Y) \tag{2.6}$$
$$\kappa = 2h/(2b + h(\mu_2 - \mu_1))$$

The matrix, which we discussed in Proposition 1, looks as follows:

$$\mathbf{A}^* = \begin{Vmatrix} 1 + \kappa\mu_1 & -\kappa\mu_2 \\ \kappa\mu_1 & 1 - \kappa\mu_2 \end{Vmatrix} \tag{2.7}$$

A check of the conditions of Proposition 1 reduces to investigating the signs of the diagonal elements of matrix (2.7) and its determinant Δ. As a result of some calculations were obtain

$$a_{ii}^* = \frac{2b + (-1)^{i+1}h(\mu_1 + \mu_2)}{2b + h(\mu_2 - \mu_1)}, \quad i = 1, 2, \quad \Delta = \det A = \frac{2b + h(\mu_1 - \mu_2)}{2b + h(\mu_2 - \mu_1)} \tag{2.8}$$

The positiveness of all three expressions of (2.8) is equivalent to the single inequality

$$\mu_1 + \mu_2 < 2\operatorname{ctg}\varphi, \quad \operatorname{ctg}\varphi = b/h \tag{2.9}$$

Note that condition (2.9) is stronger than that obtained by Painlevé [1] (assuming the inequality $\mu_2 > \mu_1$)

$$\mu_2 - \mu_1 < 2\operatorname{ctg}\varphi \tag{2.10}$$

To clarify this disagreement we must bear in mind that Painlevé [1] confined himself to the case $Y = 0$. It then follows from Eqs (2.4) that $N_1 = N_2$, i.e. $x_1 = x_2$ and $y_1 = y_2$ in formulae (2.6). The mapping (2.6) reduces to the one-dimensional mapping

$$x = (1 + (\mu_1 - \mu_2)\kappa)y = y\Delta \tag{2.11}$$

As it applies to the mapping (2.11) the condition of Proposition 1 reduces to the inequality

$$|\mu_2 - \mu_1| < 2\operatorname{ctg}\varphi$$

which agrees with Painlevé's result (2.10).

Note that the condition for there to be no paradoxes (2.9) also remains true in the case when, at the initial instant of time, the rod moves to the left. This can be shown by turning Fig. 1 through 180° and interchanging the coefficients μ_1 and μ_2.

2. If both constraints are unilateral, i.e. $q_1 \geqslant 0, q_2 \geqslant 0$, then $N_1 \geqslant 0, N_2 \geqslant 0$. In this case $|N| = N$, and the additional variables are defined by formulae (1.6). The elements of the matrix A^* are the coefficients of N in Eqs (2.5) and take the form

$$k^2 a_{ii}^* = k^2 + b^2 + (-1)^i \mu_i bh, \ i = 1, 2; \ k^2 a_{ij}^* = -k^2 + b^2 + \mu_i bh, \ i \neq j; \quad i, j = 1, 2$$

The conditions of Proposition 1 reduce to the inequalities

$$\mu_1 bh < k^2 + b^2, \quad -\mu_2 bh < k^2 + p^2, \quad \mu_1 - \mu_2 < 2b/h \tag{2.12}$$

3. One of the constraints is unilateral while the second is bilateral: $q_1 \geqslant 0, q_2 \equiv 0$. In system (2.5) we put

$$x_1 = \ddot{q}_1, \ y_1 = N_1, \ x_2 = (|N_2| + N_2)/2, \ y_2 = (|N_2| - N_2)/2, \ Y = 0, \ M = 0$$

and express the variables x_1, x_2 in terms of y_1, y_2

$$x_1(k^2 + b^2 + \mu_2 bh) = 2by_1(2b + h(\mu_2 - \mu_1)) + 4\mu_2 bhy_2$$

$$x_1(k^2 + b^2 + \mu_2 bh) = y_1(k^2 - b^2 + \mu_1 bh) + y_2(k^2 + b^2 - \mu_2 bh)$$

The conditions of Proposition 1 take the form

$$\mu_2 |b| h < k^2 b^2 (\mu_1 - \mu_2) bh < 2b^2 (\mu_1 + \mu_2) bh < 2b^2 \tag{2.13}$$

4. In the case when $q_1 \equiv 0, q_2 \geqslant 0$, which is the opposite of case 3, it is sufficient to replace b by $-b$ in conditions (2.13) and interchange the coefficients μ_1 and μ_2. The conditions for there to be no paradoxes take the form

$$\mu_1 |b| h < k^2 + b^2 (\mu_1 - \mu_2) bh < 2b^2 - (\mu_1 + \mu_2) bh < 2b^2 \tag{2.14}$$

Regions (2.9), (2.12), (2.13) and (2.14) are constructed in the plane of the parameters (μ_1 and μ_2) in Fig. 1(b) for the case when $b > k$. The set of solutions of inequality (2.9) represents an isosceles right triangle with the leg of the triangle $\alpha = 2b/h$, while conditions (2.12) and (2.14) represent one and the same rectangular trapezium with height $\beta = (k^2 + b^2)/bh$ and bases α and $\alpha + \beta$, parallel to the ordinate axis. The region (2.13) is a trapezium of height β and bases α and $\alpha - \beta$, parallel to the abscissa axis. In the case when $0 < b < k$ the first trapezium contains the triangle while region (2.13) coincides with (2.10).

We can similarly interpret the regions where there are no paradoxes in the case when $b < 0$, for which it is sufficient to interchange the coefficients μ_1 and μ_2 while simultaneously changing the sign of b.

3. CLASSIFICATION OF THE PARADOXES IN THE CASE WHEN $k \leqslant 2$

We will discuss typical paradoxes in mechanical systems with one or two pairs of Coulomb friction. In the simplest case when $k = 1$, the defining mapping (1.7) is a piecewise-linear function of the same variable. The graph consists of two rays converging at the origin of coordinates (Fig. 2). If the coefficients A_1 and A_2 in (2.3) have the same signs, the graphs intersect the z axis, whence it follows that the defining mapping and the absence of paradoxes are mutually equivalent (the continuous line in Fig. 2). If $A_1 A_2 < 0$, the rays lie on the same side of the z axis and, depending on the sign of B, the mapping has two or none originals (the dashed line in Fig. 2). Such a situation is called a fold the theory of singularities.

We will now consider the case when $k = 2$. We will assume all the principal minors of the matrix A^* are non-zero, in which case the number of solutions of Eq. (2.3) for any right-hand side is finite [9]. The conditions of Proposition 1 in the case when $k = 2$ reduces to the following set of three inequalities

$$a_{11}^* > 0, \quad a_{22}^* > 0, \quad \Delta = \det A^* > 0 \tag{3.1}$$

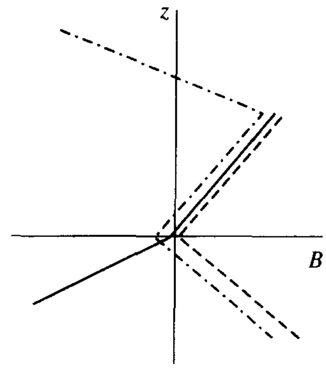

Fig. 2

Conditions (3.1) guarantee the existence of a unique solution of system (2.3) for any vector **B**.

The non-fulfilment of a least one of inequalities (3.1) indicates a paradoxical situation when the solution is non-unique or does not exist for certain **B**. We multiply both sides of Eq. (2.3) by the matrix $\mathbf{A}^{-1}$

$$\mathbf{x} - \mathbf{A}^*\mathbf{y} = \mathbf{S}^*(\mathbf{z}) = \mathbf{B}^*, \ \mathbf{B}^* = \mathbf{A}^{-1}\mathbf{B} \tag{3.2}$$

To analyse possible cases we will construct transforms of each quadrant of the coordinate z-plane into a B-plane due to the action of the mapping, defined by the left-hand side of formula (3.2). Since this mapping is piecewise-linear, the transforms of the coordinate axes are lines having a discontinuity at the origin of coordinates. We will denote the coordinate unit vectors in the z- and B-planes by $e_{1,2}$ and $e'_{1,2}$ respectively; then

$$\mathbf{S}^*(\mathbf{e}_1) = \mathbf{e}'_1, \ \mathbf{S}^*(\mathbf{e}_2) = \mathbf{e}'_2, \ \mathbf{s}_1 = \mathbf{S}^*(-\mathbf{e}_1) = -a^*_{11}\mathbf{e}'_1 - a^*_{21}\mathbf{e}'_2$$
$$\mathbf{s}_2 = \mathbf{S}^*(-\mathbf{e}_2) = -a^*_{12}\mathbf{e}'_1 - a^*_{22}\mathbf{e}'_2$$

Consequently, $\mathbf{S}^*(L_1) = L'_1$ (we denote the jth quadrants of the coordinate z-plane and B-plane by L_j and L'_j, respectively), the region $\mathbf{S}^*(L_2)$ is bounded by the half-lines with direction vectors e'_2 and $\mathbf{s}_1$, $\mathbf{S}^*(L_4)$ lies between e'_1 and $\mathbf{s}_2$, $\mathbf{S}^*(L_3)$ lies between $\mathbf{s}_1$ and $\mathbf{s}_2$.

The location of the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ in the B-plane depends on the elements of the matrix $\mathbf{A}^*$ and, in turn, determines the nature of the singularity of the mapping $\mathbf{S}^*$. In the regular case (3.1) the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ lie outside the first quadrant and form a right system, and hence the sets $\mathbf{S}^*(L_j)(j = 1, 2, 3, 4)$ do not intersect pairwise, and their sum comprises the whole B-plane.

In Fig. 3 we show possible forms of the mapping $\mathbf{S}^*$.

Case a

$$a^*_{11} > 0, \quad a^*_{22} > 0, \quad \Delta < 0, \quad a^*_{12} < 0$$

(consequently, $a^*_{21} < 0$). Here $\mathbf{s}_1 \in L'_2$, $\mathbf{s}_2 \in L'_4$, and the system of these two vectors has a left orientation (Fig. 3a). Each point lying inside the obtuse angle between the half-lines and the direction vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ has two originals, and none outside this angle. In the theory of singularities, such a situation is called a fold in this context the fold line is a dashed line.

Case b

$$a^*_{11} > 0, \quad a^*_{22} > 0, \quad \Delta < 0, \quad a^*_{12} > 0$$

(consequently, $a^*_{21} > 0$). Here $\mathbf{s}_1 \in L'_3$, $\mathbf{s}_2 \in L'_3$, and the system of these two vectors, as before, has a left orientation (Fig. 3b). Points lying inside the angle formed by these vectors have three originals, and outside it one original. Such a situation is called a cusp.

Case c

$$a^*_{11} < 0, \quad a^*_{22} > 0, \quad \Delta > 0, \quad a^*_{12} < 0 \tag{3.3}$$

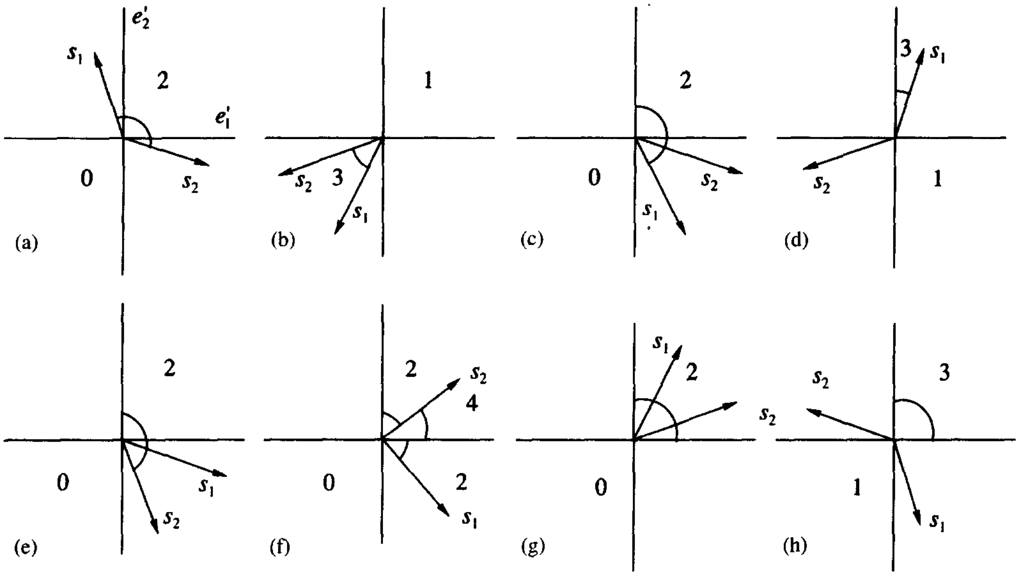

Fig. 3

(consequently, $a_{21}^* > 0$). Here we have $s_1 \in L_4'$, $s_2 \in L_4'$ (a right orientation, Fig. 3c). Points lying inside the angle by the vectors s_1 and e_2' have two originals, and none outside this angle (a fold).

Case d

$$a_{11}^* < 0, \quad a_{22}^* > 0, \quad \Delta > 0, \quad a_{12}^* > 0 \tag{3.4}$$

(consequently, $a_{21}^* < 0$). Here $s_1 \in L_1'$, $s_2 \in L_3'$ (a right orientation, Fig. 3d). Points lying inside the angle formed by the vectors s_1 and e_2' have three originals, and outside this angle one original (a cusp).

Case e

$$a_{11}^* < 0, \quad a_{22}^* > 0, \quad \Delta < 0$$

The vector s_1 lies in the right half-plane while s_2 lies in the lower half-plane (left orientation, in Fig. 3e we show the subcase $a_{21}^* > 0$, if $a_{21}^* < 0$, then $s_1 \in L_1'$). Points lying outside the angle formed by the vectors s_1 and e_2' (the angle is drawn in the positive direction from the first vector to the second and can be greater than the expanded form), have two originals, and none outside this angle (a fold).

Case f

$$a_{11}^* < 0, \quad a_{22}^* < 0, \quad \Delta > 0$$

The vector s_1 then lies in the right half-plane while s_2 lies in the upper plane (right orientation, in Fig. 3f we show one of the four possible subcases $a_{21}^* > 0, a_{21}^* < 0$). Points of the B-plane in this case can have four, two or none originals, which characterizes a double fold (the form of this singularity can be represented by folding a sheet of paper into four so that the line of the second fold is not perpendicular to the line of the first fold).

Case g

$$a_{11}^* < 0, \quad a_{22}^* < 0, \quad \Delta < 0, \quad a_{12}^* > 0$$

(consequently, $a_{21}^* < 0$), then $s_{1,2} \in L_1'$ (left orientation, Fig. 3g). Points lying in the first quadrant, have two originals, while the remaining points have none (a fold).

Case h

$$a_{11}^* < 0, \quad a_{22}^* < 0, \quad \Delta < 0, \quad a_{12}^* > 0$$

(consequently, $a^*_{21} > 0$). We have $\mathbf{s}_1 \in L'_4$, $\mathbf{s}_2 \in L'_2$ (left orientation, Fig. 4h). Points lying in the first quadrant have three originals, while the remaining points of the plane have a single original (a cusp).

Remark 1. In Cases c–e the inequality $a^*_{11}a^*_{22} < 0$ is satisfied. In this case, it was assumed above that $a^*_{11} < 0$, but the conclusions drawn on the qualitative type of singularity also remain true for $a^*_{22} < 0$ if we replace a^*_{12} and a^*_{21} in inequalities (3.3) and (3.4). We can convince ourselves of the correctness of this observation by changing the numbering of the variables q_1 and q_2.

Remark 2. The problem of resolving the paradoxical situations lies within the framework of the initial formulation of the problem and requires the inclusion of additional physical considerations (see, for example, [12–17]). Note, however, that the above-mentioned singularities cannot be eliminated by correcting the Coulomb friction laws: the presence of a break in the characteristic ensures that the type of singularity is preserved (Figs 2 and 3) at the origin of coordinates. Moreover, a "corrected" friction law may lead to the occurrence of additional singularities. For example, a case was considered in [17] when the friction coefficient decreases as the normal load increases. The corresponding bifurcation diagram is shown schematically in Fig. 2 by the dash–dot line: here there are two folds. As a result the paradox of the non-existence of solutions is eliminated, but the non-uniqueness remains (in this case, in a certain range of values of B, the number of solutions increases to three).

Example. We will discuss the nature of the possible paradoxes in the Painlevé–Klein example, considered in the previous section. We will confine ourselves to the classical formulation of this problem, assuming both constraints to be bilateral. As follows from Eqs (2.8), the relation $a^*_{11}a^*_{22} < 0$ is equivalent to condition (2.9), where the inequality sign is reversed. Moreover, the inequality $\Delta < 0$ is equivalent to condition (2.10), where both sides are taken in absolute value. Hence, in the plane of the parameters (μ_1, μ_2) there are three regions in which conditions (3.1) for the existence of a unique solution (Fig. 1c) break down. In this case regions 1 and 3 correspond to "fold" type discontinuities (case e) while region 2 corresponds to a cusp singularity (case d).

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